Anna Kamińska

Some convexity properties of Musielak-Orlicz spaces of Bochner type

In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the 13th Winter School on Abstract Analysis, Section of Analysis. Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 10. pp. [63]–73.

Persistent URL: http://dml.cz/dmlcz/701863

SOME CONVEXITY PROPERTIES OF MUSIELAK-ORLICZ SPACES OF BOCHNER TYPE

A. Kamińska

Abstract. It is shown here that if a Banach space X and Musielak-Orlicz space L_φ are both locally uniformly rotund or uniformly rotund in every direction then the space $L_\varphi(X)$ of Bochner type has the same properties. Moreover criteria for these properties have been given for a subspace of finite elements $E_\varphi(X)$.

Introduction. Many authors have been examined the question whether a geometrical property lifts from a Banach space X to the Lebesque-Bochner space $L^p(X)$. M.Smith in [7] has given a brief survey of those problem. Similar questions have also been considered for Orlicz or Musielak-Orlicz space. H.Hudzik in [4] has been shown that if X and Musielak-Orlicz space L_φ are both uniformly rotund then $L_\varphi(X)$ is also uniformly rotund. N.Herrndorf in [3] has proved that Bochner-Orlicz space $L_\varphi(X)$ is locally uniformly rotund iff both X and L_φ have this property. Here we consider two geometrical properties: local uniform rotundity (LUR) and uniform rotundity in every direction (URED), in the context of Musielak-Orlicz spaces of vector functions. In paper [5] there have been presented criteria for the above properties in Musielak-Orlicz spaces of scalar functions L_φ, expressed in terms of function φ. Here it is shown that Musielak-Orlicz space $L_\varphi(X)$ of Bochner type is LUR(URED) iff both X and L_φ are LUR(URED). Similar results are also shown for the subspace of finite elements $E_\varphi(X)$ of the space $L_\varphi(X)$. Subspaces of this kind play an important role in the theory of spaces of Orlicz type.

Since the paper is a direct continuation and generalization of results from [5], we refer a reader to those paper for basic notations and definitions as well as for some Lemmas and Theorems. Now, we give some additional notations and definitions. For $u,v \in \mathbb{R}$, let us denote $\max(u,v) = u \vee v$, $\min(u,v) = u \wedge v$. The Musielak-Orlicz

This paper is in final form and no version of it will be submitted for publication elsewhere.

space $L_\varphi(X)$ of Bochner type is a family of all strongly measurable functions $x : T \to X$ such that $I_\varphi(\lambda x) = \int_T \varphi(\lambda \|x(t)\|, t)d\mu < \infty$ for some $\lambda > 0$ dependent on x, where X is a Banach space. The space $L_\varphi(X)$ is equipped with Luxemburg norm. The subspace of finite elements $E_\varphi(X)$ is a family of all strongly measurable functions x such that $I_\varphi(\lambda x) < \infty$ for every $\lambda > 0$. Suppose in the following that measure μ is σ-finite. There exists an increasing sequence (T_i) such that $\mu T_i < \infty$, $\mu(T \setminus \bigcup_{i=1}^{\infty} T_i) = 0$ and

$$(0.1) \qquad \sup_{t \in T_i} \varphi(u,t) < \infty$$

for every $u \in \mathbb{R}_+$ and $i \in \mathbb{N}$. Indeed, let (A_i) be a sequence of pairwise disjoint sets such that $\mu A_i < \infty$ and $\mu(T \setminus \bigcup_{i=1}^{\infty} A_i) = 0$. Let $A_{nm}^i = \{t \in A_i : \varphi(n,t) \leq m\}$. Since $\bigcup_{m=1}^{\infty} A_{nm}^i = A_i$, $\mu(A_i \setminus A_{nm}^i) \to 0$ as $m \to \infty$. Therefore, for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists m_n such that $\mu(A_i \setminus A_{nm_n}^i) < \varepsilon/2^n$. Hence $\mu(A_i \setminus \bigcap_{n=1}^{\infty} A_{nm_n}^i) \leq$ $\leq \sum_{n=1}^{\infty} (A_i \setminus A_{nm_n}^i) < \varepsilon$. Denoting $B_\varepsilon^i = \bigcap_{n=1}^{\infty} A_{nm_n}^i$ we have $\sup_{t \in B_\varepsilon^i} \varphi(n,t) < \infty$ for every $i, n \in \mathbb{N}$. Let us take a sequence $(B_{\varepsilon_j}^i)_{i,j}$ where (ε_j) is a sequence tending to zero. So, we have

$$\mu\left(T \setminus \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} B_{\varepsilon_j}^i\right) \leq \sum_{i=1}^{\infty} \mu\left(A_i \setminus \bigcup_{j=1}^{\infty} B_{\varepsilon_j}^i\right) = 0,$$

because $\mu(A_i \setminus \bigcup_{j=1}^{\infty} B_{\varepsilon_j}^i) \leq \mu(A_i \setminus B_{\varepsilon_j}^i) \leq \varepsilon_j$ for all $j \in \mathbb{N}$.

Finally, we transform the sequence $(B_{\varepsilon_j}^i)_{i,j}$ into (T_i) with desired properties.

In virtue of (0.1) it is seen that $E_\varphi(X)$ is always nonempty, because all characteristic functions of T_i belong to E_φ. Condition (0.1) has appeared in [2], in the context of decomposability of the subspace of finite elements, but the author has not given a proof.

For a Banach space $(X, \|\ \|)$ we define the following moduli of rotundity

$$\delta(y,\varepsilon) = \inf\{1 - \|(x+y)/2\| : \|x\| \leqslant 1, \quad \|x-y\| \geqslant \varepsilon\}$$

for $\|y\| = 1$, and

$$\delta(\varepsilon, \rightarrow z) = \inf\{1 - \|((x+\lambda z)+x)/2\| : \|x\| \leqslant 1, \|x+\lambda z\| \leqslant 1, \|\lambda z\| \geqslant \varepsilon\}$$

for $z \neq 0$. The space $(X, \|\ \|)$ is LUR(URED) iff $\delta(y,\varepsilon) > 0$ $(\delta(\varepsilon, \rightarrow z) > 0)$ for every $\varepsilon > 0$ and every y belonging to a unit sphere of X (every $0 \neq z \in X$) [1] . If X is separable and y strongly measurable function then compositions $\delta(y(t),\varepsilon), \delta(\varepsilon, \rightarrow y(t))$ are measurable functions. It is trivial to check that Theorem 0.2, Lemmas 0.3, 0.4, 4 in [5] are also true for the space $L_\varphi(X)$ of Bochner type.

For arbitrary $x, y \in X$ we have $\|x+y\| \vee \|x\| \geqslant \|x \mp y\| \geqslant \|y\| - \|x\| \geqslant \|y\| - (\|x+y\| \vee \|x\|)$. It implies that

(0.2) $\qquad \|x+y\| \vee \|x\| \geqslant \|y\|/2$

for every $x, y \in X$. This simple inequality plays a similar role to Lemma 2 in [5] .

Results.

1. Lemma. If φ doesn't satisfy condition Δ_2 then there exists a sequence $(y_k) \subset E_\varphi$ such that $I_\varphi(y_k) \longrightarrow 0$ and $\|y_k\|_\varphi \longrightarrow 1$ and $\mu\left(T \setminus \bigcup_{k=1}^{\infty} \operatorname{supp} y_k\right) > 0$.

Proof. It is easily seen that condition Δ_2 is fulfilled iff $\int_T h_n(t)\,d\mu < \infty$ for some $n \in \mathbb{N}$, where

$$h_n(t) = \sup_{u \in \mathbb{R}}\{\varphi((1+\tfrac{1}{n})u,t) - 2^n\varphi(u,t)\}.$$

Let (u_i) be the set of rational numbers and

$$A_{nmi} = \{t \in T_m : \varphi((1+\tfrac{1}{n})u_i,t) \geqslant 2^n\varphi(u_i,t)\},$$

where (T_m) is a sequence from condition (0.1). Putting $\{x_{jn}(t)\}_j = \{u_i\,\chi_{A_{nmi}}(t)\}_{m,i}$ and

$$g_n(t) = \sup_{u \in \mathbb{R}}\{\varphi((1+\tfrac{1}{n})u,t) : \varphi((1+\tfrac{1}{n})u,t) \geqslant 2^n\varphi(u,t)\}$$

we get

$$g_n(t) = \sup_{i,n} \varphi((1+\tfrac{1}{n})u_i\,\chi_{A_{nmi}}(t),t) = \sup_j \varphi((1+\tfrac{1}{n})x_{jn}(t),t).$$

It is evident, that $x_{jn} \in E_\varphi$ for each $j, n \in \mathbb{N}$. If condition Δ_2

is not satisfied then

$$\int_T g_n(t)\,d\mu = \infty$$

for each $n \in \mathbb{N}$, because $g_n(t) \geq h_n(t)$. Putting $g_{nl}(t) = \max_{1\leq j\leq l} \varphi\left(\left(1 + \frac{1}{n}\right)x_{jn}(t), t\right)$ we have $g_{nl}(t) \uparrow g_n(t)$ as $l \to \infty$ and hence

$$\int_T g_{nl(n)}(t)\,d\mu \geq 2^n \tag{1.1}$$

for every $n \in \mathbb{N}$ and some $l(n) \in \mathbb{N}$. Denoting $\bar{x}_n(t) = \max_{1\leq j\leq l(n)} x_{jn}(t)$, we have

$$g_{nl(n)}(t) = \varphi\left(\left(1 + \frac{1}{n}\right) \bar{x}_n(t), t\right).$$

We find an increasing subsequence $(n_k) \subset \mathbb{N}$ and a sequence (A_k) of pairwise disjoint sets such that

$$\int_{A_k} \varphi\left(\left(1 + \frac{1}{n_k}\right)\bar{x}_{n_k}(t), t\right)d\mu = 1 \tag{1.2}$$

for each $k \in \mathbb{N}$, by condition (1.1) and Lemma 1.7$_\infty$3 in [6]. We can take a sequence (A_k) in such a way that $\mu\left(T \setminus \bigcup_{k=1}^{\infty} A_k\right) > 0$. Moreover, we get

$$\varphi\left(\left(1 + \frac{1}{n}\right) \bar{x}_n(t), t\right) \geq 2^n \varphi(\bar{x}_n(t), t) \tag{1.3}$$

for each $n \in \mathbb{N}$, by definition of sets A_{nmi} and functions $\bar{x}_n$. Let us put

$$y_k(t) = \bar{x}_{n_k}(t)\, \chi_{A_k}(t).$$

It is evident that $y_k \in E_\varphi$. Moreover

$$\begin{aligned} I_\varphi(y_k) &= \int_{A_k} \varphi(\bar{x}_{n_k}(t), t)\, d\mu \\ &\leq 1/2^{n_k} \int_{A_k} \varphi\left(\left(1 + \frac{1}{n_k}\right) \bar{x}_{n_k}(t), t\right)d\mu \\ &= 1/2^{n_k} \longrightarrow 0, \end{aligned}$$

as $k \to \infty$, by (1.3) and (1.2). But $I_\varphi\left(\left(1 + \frac{1}{n_k}\right)y_k\right) = \int_{A_k} \varphi\left(\left(1 + \frac{1}{n_k}\right) \bar{x}_{n_k}(t), t\right)d\mu = 1$ for each $k \in \mathbb{N}$. Hence

$$\|y_k\|_\varphi = 1/(1 + (1/n_k)) \xrightarrow[\infty]{} 1,$$

as $k \to \infty$. This ends the proof, because $\bigcup_{k=1}^{\infty} \operatorname{supp} y_k = \bigcup_{k=1}^{\infty} A_k$.

<u>2.Lemma.</u> If X is locally uniformly rotund, $\varphi(\cdot, t)$ is strictly convex for $t \in T \setminus T_0$, where T_0 is some null set, then for every $\varepsilon, \alpha_1, \alpha_2 \in (0,\infty)$, $p \in (0,1)$ there exists a measurable function $q : T \longrightarrow (0,1)$ such that

$$\varphi(\|(u + v)/2\|, t) \leq (1 - q(t))\,(\varphi(\|u\|, t) + \varphi(\|v\|, t))/2$$

for all $t \in T \setminus T_0$ and all $u, v \in X$ satisfying the following conditions $\|u - v\| \geq \varepsilon(\|u\| \vee \|v\|)$, $\varphi(\|u\| \vee \|v\|, t) \in [\alpha_1, \alpha_2]$, $v \neq 0$ and $\delta(v/\|v\|, \varepsilon/2) \geq p$.

<u>Proof.</u> Let ε_1 be some fixed positive number such that

$$(2.1) \qquad \varepsilon_1 \leq p/(2-p) \wedge p/(1+p) \wedge \varepsilon/2 .$$

If $|\|u\| - \|v\|| \geq \varepsilon_1(\|u\| \vee \|v\|)$ then applying Lemma 1 of [5] we get the desired inequality with some function $q_1 : T \to (0,1)$.

Suppose then $|\|u\| - \|v\|| < \varepsilon_1(\|u\| \vee \|v\|)$. We consider two cases. If $\|u\| \leq \|v\|$ then

$$(1 - \varepsilon_1)\|v\| < \|u\| \leq \|v\| \quad \text{and} \quad \|u/\|v\| - v/\|v\|\| \geq \varepsilon ,$$

by our assumptions. Hence and by the local uniform rotundity of X and by (2.1) it holds

$$\begin{aligned}(2.2) \quad \|(u + v)/2\| &\leq (1 - \delta(v/\|v\|, \varepsilon))\|v\| \\ &< (1 - p)/(1 - \varepsilon_1)(\|u\| + \|v\|)/2 \\ &\leq (1 - p)/(1 - p/(2-p))(\|u\| + \|v\|)/2 \\ &= (1 - p/2)(\|u\| + \|v\|)/2.\end{aligned}$$

If $\|u\| > \|v\|$ then we have

$$(2.3) \qquad (1 - \varepsilon_1)\|u\| \leq \|v\| < \|u\|.$$

Moreover

$$\begin{aligned}\varepsilon &\leq \|u/\|u\| - v/\|u\|\| \leq \|u/\|u\| - v/\|v\|\| + (1 - \|v\|/\|u\|) \\ &\leq \|u/\|u\| - v/\|v\|\| + \quad /2,\end{aligned}$$

by the assumption $\|u - v\| \geq \varepsilon(\|u\| \vee \|v\|)$ and inequalities (2.1) and (2.3). Then $\|u/2\|u\| + v/2\|v\|\| \leq 1 - \delta(v/\|v\|, \varepsilon/2)$. Hence and by (2.4) and (2.1) we obtain

$$\begin{aligned}(1/\|v\|)\|(u + v)/2\| &= \|(\|u\|/\|v\|)u/2\|u\| + v/2\|v\|\| \\ &\leq \|u/2\|u\| + v/2\|v\|\| + (1/2)(\|u\|/\|v\| - 1) \\ &\leq 1 - \delta(v/\|v\|, \varepsilon/2) + \varepsilon_1/2(1 - \varepsilon_1) \\ &\leq 1 - p + (p/(1+p))/2(1 - p/(1+p)) \\ &= 1 - p/2,\end{aligned}$$

since the function $\varepsilon_1 \longmapsto \varepsilon_1/2(1 - \varepsilon_1)$ is nondecreasing. Hence and by $\|u\| > \|v\|$ we get inequality (2.2) immediately. Now, it is enough to apply the convexity of φ, to get thesis of the lemma with the function $q(t) = \min(q_1(t), p/2)$.

<u>3. Lemma.</u> If X is uniformly rotund in every direction, $\varphi(\cdot, t)$ is strictly convex for $t \in T \setminus T_0$, where T_0 is some null set, then for every $\varepsilon, \alpha_1, \alpha_2 \in (0, \infty)$, $p \in (0,1)$ there exists a measurable function $q: T \to (0,1)$ such that

$$\varphi(\|u + v/2\|, t) \leq (1 - q(t))(\varphi(\|u + v\|, t) + \varphi(\|u\|, t))/2$$

for all $t \in T \setminus T_0$ and every $u, v \in X$ satisfying the following conditions

$\|v\| \geqslant \varepsilon(\|v+u\| \vee \|u\|)$, $\varphi(\|v+u\| \vee \|u\|, t) \in [\alpha_1, \alpha_2]$, $v \neq 0$ and $\delta(\varepsilon, \to v) \geqslant p$.

Proof. Let $\varepsilon_1 = p/(2-p)$. If $|\|v+u\| - \|u\|| > \varepsilon_1(\|v+u\| \vee \|u\|)$ then by Lemma 1 of [5] we get immediately the desired inequality with some function q_1 dependent on p, α_1, α_2.

Let now $|\|v+u\| - \|u\|| \leqslant \varepsilon_1(\|v+u\| \vee \|u\|)$. It implies the following inequality

(3.1) $(1-\varepsilon_1)(\|v+u\| \vee \|u\|) \leqslant \|v+u\| \wedge \|u\|$.

Without loss of generality we can put $\|v+u\| \vee \|u\| > 0$. Since $\|v\|/(\|v+u\| \vee \|u\|) \geqslant \varepsilon$, $\|u\|/(\|v+u\| \vee \|u\|) \leqslant 1$, $\|u+v\|/(\|v+u\| \vee \|u\|) \leqslant 1$ and by definition of the modulus $\delta(\varepsilon, \to v)$ we get

(3.2) $\|((u+v)+u)/2\| \leqslant (1-\delta(\varepsilon, \to v))(\|v+u\| \vee \|u\|)$.

But

$$\|v+u\| \vee \|u\| \leqslant 1/(1-\varepsilon_1)(\|v+u\| \vee \|u\| + \|v+u\| \wedge \|u\|)/2$$
$$= 1/(1-\varepsilon_1)(\|u\| + \|u+v\|)/2,$$

by inequality (3.1). Taking into consideration in (3.2) that $\delta(\varepsilon, \to v) \geqslant p$ and $\varepsilon_1 = p/(2-p)$ we get $\|u+v/2\| \leqslant (1-p/2)(\|u\| + \|v+u\|)/2$. Applying the convexity of φ we obtain the thesis with $q(t) = \min(q_1(t), p/2)$.

Proposition. If φ doesn't fulfil condition Δ_2 then E_φ is not locally uniformly rotund and it is not uniformly rotund in every direction.

Proof. Let $(y_n) \subset E_\varphi$ be a sequence from Lemma 1 i.e. $I_\varphi(y_n) \to 0$ and $\|y_n\|_\varphi \to 1$ and $\mu(T \setminus \bigcup_{n=1}^{\infty} \operatorname{supp} y_n) > 0$. There exists a set A of positive measure such that $A \subset (T \setminus \bigcup_{n=1}^{\infty} \operatorname{supp} y_n) \cap T_m$ for some $m \in \mathbb{N}$. We have $I_\varphi(u\chi_A) < \infty$ for each $u \geqslant 0$, by (0.1). Since a function $u \to I_\varphi(u\chi_A)$ is convex and finite, it is continous and $\lim_{u\to\infty} I_\varphi(u\chi_A) = \infty$. Therefore there exist u_1, u_2 and u_n such that

$$I_\varphi(u_1\chi_A) = 1, \quad I_\varphi(u_2\chi_A) = 1/2, \quad I_\varphi(u_n\chi_A) = 1 - I_\varphi(y_n).$$

Let us put

$$z_1(t) = u_1\chi_A(t), \qquad z_2(t) = u_2\chi_A(t),$$
$$z_{1n}(t) = u_n\chi_A(t) + y_n(t), \qquad z_{2n}(t) = y_n(t).$$

The above all functions belong to E_φ, by (0.1). We have $I_\varphi(z_1) = 1$ and $I_\varphi(z_2) = 1/2$. Hence $\|z_1\|_\varphi = 1$ and $\|z_2\|_\varphi < 1$. We have also $I_\varphi(z_{1n}) = 1$, $I_\varphi(z_{2n}) \to 0$ and $\|z_{2n}\|_\varphi \to 1$. Since $|z_1(t) - z_{1n}(t)| \geqslant |y_n(t)|$, so $\|z_1 - z_{1n}\|_\varphi \geqslant \|y_n\|_\varphi \to 1$. However $\|(z_1 + z_{1n})/2\|_\varphi \geqslant I_\varphi((z_1 + z_{1n})/2) = I_\varphi(((u_1 + u_n)/2)\chi_A) + I_\varphi(y_n/2) \geqslant I_\varphi(u_n\chi_A) = 1 - I_\varphi(y_n) \to 1$, as $n \to \infty$. This shows that E_φ is not locally uniformly rotund.

Taking into consideration z_2 and z_{2n} we obtain $I_\varphi(z_2 + z_{2n}) = I_\varphi(z_2) + I_\varphi(z_{2n}) < 1$ for sufficiently large n. Hence $\|z_2 + z_{2n}\|_\varphi \leqslant 1$. But, the inequality $|z_2(t)/2 + z_{2n}(t)| = (1/2)|z_2(t)| + |z_{2n}(t)| \geqslant |z_{2n}(t)|$ implies $\|z/2 + z_{2n}\|_\varphi \geqslant \|z_{2n}\|_\varphi \to 1$, as $n \to \infty$. It shows that E_φ is not uniformly rotund in every direction and ends the proof.

<u>Theorem.</u> If X is separable then the following conditions are equivalent

(1) $L_\varphi(X)$ is LUR (URED),

(2) X and L_φ are LUR (URED),

(3) $E_\varphi(X)$ is LUR (URED),

(4) X and E_φ are LUR (URED),

(5) the function φ is strictly convex and satisfies condition Δ_2 and X is LUR (URED).

<u>Proof.</u> Implications (1)$\to$(2) and (3)$\to$(4) are immediate, because X and L_φ or E_φ are isometric subspaces of $L_\varphi(X)$ or $E_\varphi(X)$ respectively. The implication (1)$\to$(3) is trivial. Implications (2)$\to$(5) and (4)$\to$(5) are results of Proposition and Theorem 0.1 in [5], because $L_\varphi = E_\varphi$ if φ satisfies condition Δ_2. So it is enough to prove that (5) implies (1). Some ideas of the proof are included in [5], but for clarity we present the investigation on the whole. First, we will show that $L_\varphi(X)$ is locally uniformly rotund. Let $\varepsilon > 0$ and $y \in L_\varphi(X)$ be such that $I_\varphi(y) = 1$. Consider the set of all x for which $I_\varphi(x) = 1$ and $I_\varphi(x - y) \geqslant \varepsilon$. By condition Δ_2 there exist $k > 0$ and a nonnegative function h, such that

(1) $\int_T h(t)\,d\mu < (1/16)\varepsilon$ and $\varphi(2u,t) \leqslant k\,\varphi(u,t) + h(t)$

for all $u \in R$ and a.e. $t \in T$. We find also constants c_1, c_2 such that $c_2 > c_1 > 1$ and

(2) $\int_{T_1} \varphi(2\|y(t)\|,t)\,d\mu < (1/32k)\varepsilon$ where

$$T_1 = \{t \in T : \varphi(\|y(t)\|,t) < 1/c_1 \vee \varphi(2\|y(t)\|,t) > c_1\}$$
and

(3) $c_1/c_2 \leqslant (1/32)(\varepsilon/k)/(k + (1/16)\varepsilon)$.

Put
$$T_x = \{t \in T : \varphi(2\|x(t)\|,t) > c_2\}.$$

Denoting $T_0(x) = T \setminus (T_1 \cup T_x)$ we have

$$T_0(x) = \{t \in T : 1/c_1 \leqslant \varphi(\|y(t)\|, t) \wedge \varphi(2\|y(t)\|, t) \leqslant c_1\} \cap \{t \in T : \varphi(2\|x(t)\|, t) \leqslant c_2\}.$$

Supposing that $I_\varphi((x-y)\chi_{T_0(x)}) < (3/4)\varepsilon$ we have $I_\varphi((x-y)\chi_{T_1 \cup T_x}) > (1/4)\varepsilon$, by the assumption $I_\varphi(x-y) \geqslant \varepsilon$. We have also

$$(4) \quad I_\varphi(y\chi_{T_1 \cup T_x}) \leqslant \int_{T_x \setminus T_1} \varphi(\|y(t)\|, t)d\mu + \int_{T_1} \varphi(\|y(t)\|, t)d\mu$$
$$\leqslant c_1 \mu(T_x \setminus T_1) + (1/32k)\varepsilon$$
$$\leqslant (c_1/c_2) \int_{T_x} \varphi(2\|x(t)\|, t)d\mu + 1/32k$$
$$\leqslant (c_1/c_2)(kI_\varphi(x) + \int_T h(t)d\mu) + (1/32k)\varepsilon$$
$$\leqslant (c_1/c_2)(k + (1/16)\varepsilon) + (1/32k)\varepsilon$$
$$\leqslant (1/16k)\varepsilon,$$

by (1), (2) and (3). Therefore

$$\varepsilon/4 < I_\varphi((x-y)\chi_{T_1 \cup T_x}) \leqslant (k/2)(I_\varphi(x\chi_{T_1 \cup T_x}) + I_\varphi(y\chi_{T_1 \cup T_x})) + \int_T h(t)d\mu \leqslant (k/2)I_\varphi(x\chi_{T_1 \cup T_x}) + (3/32)\varepsilon.$$

Hence $I_\varphi(x\chi_{T_1 \cup T_x}) \geqslant (5/16k)\varepsilon$. This fact joined with (4) gives an inequality $I_\varphi(y\chi_{T_0(x)}) - I_\varphi(x\chi_{T_0(x)}) > (1/4k)\varepsilon$. Now, applying Lemma 0.4 from [5] there exists a positive number α dependent only on ε, k such that

$$(5) \qquad I_\varphi((x-y)\chi_{T_0(x)}) \geqslant \alpha,$$

for every considered x. Denote $\nu_x(A) = I_\varphi((x-y)\chi_{A \cap T_0(x)})$. These set functions satisfy assumptions of Lemma 3 in [5], if we put x in place of τ. Indeed, $\nu_x(A) \leqslant (1/2)I_\varphi(2x\chi_{A \cap T_0(x)}) + (1/2)I_\varphi(2y\chi_{A \cap T_0(x)}) \leqslant ((c_1+c_2)/2)\mu A$. It implies that $\nu_x(A) \leqslant \varepsilon$ if $\mu A \leqslant 2\varepsilon/(c_1+c_2)$. Moreover $\mu(T \setminus T_1) < \infty$ and $\nu_x(T_1) = 0$. Putting $T_\varepsilon = T \setminus T_1$ we showed the first assumption of the lemma. The second assumption is obvious by [5]. Therefore, taking the function $\delta(y(t)/\|y(t)\|, \alpha/16)$ as $q(t)$ in this lemma, there exists $p > 0$ such that

$$(6) \qquad I_\varphi((x-y)\chi_{T_0(x) \cap T_0'}) \geqslant (3/4)\alpha,$$

where $T_0' = \{t \in T : \delta(y(t)/\|y(t)\|, \alpha/16) \geqslant p\}$.

Now, let $q(t)$ be the function from Lemma 2 chosen for $\alpha/16$, $1/c_1$, c_2', p in place of ε, α_1, α_2, p. Applying again Lemma 3 in the context of (6), there exists a constant $q \in (0,1)$ such that

(7) $$I_\varphi((x-y)\chi_{T_0(x)\cap T_0'\cap T_0''}) \geq \alpha/2,$$

where $T_0'' = \{t \in T : q(t) \geq q\}$. Denoting $U(x) = T_0(x)\cap T_0'\cap T_0''$, let $T_2(x) = \{t \in U(x) : \|x(t) - y(t)\| \geq (\alpha/8)(\|x(t)\| \vee \|y(t)\|)\}$. If $t \in T_2(x)$ then values $x(t)$ and $y(t)$ satisfy assumptions of Lemma 2 and so

$$\varphi(\|(x(t)+y(t))/2\|,t) \leq (1-q)(\varphi(\|x(t)\|,t) + \varphi(\|y(t)\|,t))/2.$$

Therefore

(8) $$I_\varphi((x+y)/2) \leq 1 - (q/2)(I_\varphi(x\chi_{T_2(x)}) + I_\varphi(y\chi_{T_2(x)})).$$

If $t \in U(x)\setminus T_2(x)$ then $\varphi(\|x(t)-y(t)\|,t) \leq (\alpha/8)(\varphi(\|x(t)\|,t) + \varphi(\|y(t)\|,t))$. Hence $I_\varphi((x-y)\chi_{U(x)\setminus T_2(x)}) \leq \alpha/4$. So, in virtue of (7) we have $I_\varphi((x-y)\chi_{T_2(x)}) > \alpha/4$. Now, we find a constant k_1 and a function h_1 such that

$$\int_T h_1(t)\,d\mu \leq \alpha/8 \quad\text{and}\quad \varphi(2u,t) \leq k\,\varphi_1(u,t) + h_1(t).$$

Then $I_\varphi(x\chi_{T_2(x)}) + I_\varphi(y\chi_{T_2(x)}) \geq (2/k_1)(I_\varphi((x-y)\chi_{T_2 x}) - \int_T h_1(t)d\mu) \geq (2/k_1)(\alpha/4 - \alpha/8) = \alpha/4k_1$. Hence and by (8) we get the following estimation

(9) $$I_\varphi((x+y)/2) \leq 1 - q\alpha/8k_1,$$

which ends the proof in virtue of Lemma 0.3 in [5].

Now, we will show that $L_\varphi(X)$ is uniformly rotund in every direction. Let $z \in L_\varphi$, $z \neq 0$ and $I_\varphi(x) \leq 1$ and $I_\varphi(x+z) \leq 1$. Assumptions of Lemma 4 in [5] are satisfied with functions z and x. Then, there exist constants $c, d, \alpha > 0$ such that

$$I_\varphi(z\chi_{W_0(x)}) > \alpha$$

for arbitrary x satisfying $I_\varphi(x) \leq 1$, where

$$W_0(x) = W_1 \cap W_x,$$

$W_1 = \{t \in T : 1/c \leq \varphi(\|z(t)\|/2,t) \text{ and } \varphi(2\|z(t)\|,t) \leq c\}$,
$W_x = \{t \in T : \varphi(2\|x(t)\|,t) \leq d\}$.

Since $z(t) \neq 0$ for every $t \in W_1$, $\delta(\varepsilon, \to z(t)) > 0$ for $t \in W_0(x)$. Moreover $\mu W_1 < \infty$. Applying Lemma 3 in [5] for set mappings $\nu_x(A) = I_\varphi(z\chi_{W_0(x)\cap A})$ and the function $\delta(\varepsilon, \to z(t))$, there exists $p \in (0,1)$ such that

$$I_\varphi(z\chi_{W_0(x)\cap W'}) \geq (3/4)\alpha,$$

where $W' = \{t \in T : \delta(\varepsilon, \to z(t)) \geqslant p\}$. Let $q(t)$ be a function from Lemma 3 chosen for constants $\alpha/8$, $1/c$, $(c+d)/2$, p taken as ε, α_1, α_2, p. Applying again Lemma 3 in [5] we obtain

$$I_\varphi(z\chi_{W_0(x)\cap W'\cap W''}) \geqslant \alpha/2 ,$$

where $W'' = \{t \in T : q(t) \geqslant q\}$ for some $q \in (0,1)$. Let

$$W_2(x) = \{t \in W(x) : \|z(t)\| \geqslant (\alpha/8)(\|z(t) + x(t)\| \vee \|x(t)\|)\}$$

where $W(x) = W_0(x) \cap W' \cap W''$. If $t \in W_2(x)$ then values of $z(t)$ and $x(t)$ satisfy assumptions of Lemma 3 with constants $\alpha/8$, $1/c$, $(c+d)/2$, p. Indeed $1/c \leqslant \varphi(\|z(t)\|/2, t) \leqslant \varphi(\|z(t) + x(t)\| \vee \|x(t)\|, t) \leqslant (c + d)/2$ for $t \in W_0(x)$, by the inequality (0.2). Therefore $\varphi(\|z(t) + x(t)/2\|, t) \leqslant (1 - q)(\varphi(\|z(t) + x(t)\|, t) + \varphi(\|x(t)\|, t))/2$ for $t \in W_2(x)$. In the sequel, proceeding similarly as in the previous proof beginning from inequality (8), we get an estimation of the type (9). So, in virtue of Lemma 0.3 in [5], the proof is finished.

<u>Remark.</u> The separability of X is used only for measurability of compositions $\delta(y(t), \varepsilon)$ and $\delta(\varepsilon, \to y(t))$. The above theorem is a generalization of some results from [3] and [8].

REFERENCES

[1] Day M.M. "Normed linear spaces", Springer-Verlag, New York, 1973.

[2] Giner E. "Espaces integraux de type Orlicz. Dualite, compacite, convergence en mesure. Applications a l'optimisation", These, Perpignan, 1977.

[3] Herrndorf N. "Local uniform convexity of Orlicz spaces of Bochner integrable functions", Preprints in Statistics no.66, University of Cologne, 1981.

[4] Hudzik H. "Uniform convexity of Musielak-Orlicz spaces with Luxemburg's norm", Comment.Math., 23(1983), 21-32.

[5] Kamińska A. "On some convexity properties of Musielak-Orlicz spaces", Supplemento ai Rendiconti del Circolo Matematico di Palermo, The 12^{th} Winter School on Abstract Analysis, Srni 1984.

[6] Kozek A. "Convex integral functionals on Orlicz spaces", Comment. Math., 21 (1) (1980), 109-135.

[7] Smith M.A. "Rotundity and extremity in $l^p(X_i)$ and $L^p(\mu, X)$", Proceedings of the M.M.Day Math. Conference, June 9-12, 1983.

[8] Smith M.A., Turett B. "Rotundity in Lebesque-Bochner function spaces", Trans.Am.Math.Soc., 251, No 1(1980), 105-118.

INSTITUTE OF MATHEMATICS
ADAM MICKIEWICZ UNIVERSITY
UL. MATEJKI 48/49 POZNAŃ
POLAND