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Remarks on powers of lattices

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REMARKS ON POWERS OF LATTICES

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A cardinal β is called an ω-power if $\beta^{\aleph_0} = \beta$. A well known result of R.S.Pierce [7] says that the power of every infinite complete Boolean algebra is an ω-power. Subsequently J.D.Monk and P.R.Sparks [6] and W.W.Comfort and A.H.Hager [2] have shown that the same is valid for σ-complete Boolean algebras. This result was improved by S.Koppelberg [4] ; she has proved that it holds for weakly-σ-complete Boolean algebras. Recently E.K. van Douwen and H.-X.Zhou [3] have obtained a topological theorem which is closely related to these results. They have proved that for every compact Hausdorff space X, the power of the lattice $L(X) = \{\mathrm{Int\,cl}\,U : U$ is a cozero-set in $X\}$ is an ω-power. Note, that the family of all regular-open subsets of a topological space X forms a complete Boolean algebra containing L(X) as an upward σ-complete sublattice, i.e. L(X) is closed under suprema of countable subsets. This leads to a natural question (see [3]) : which lattices have power being ω-power ? Concerning this question I have obtained in [1] the following results :

<u>Theorem</u> 1. There exists an upward σ-complete sublattice L of a complete Boolean algebra such that $|L|$ is not ω-power.

In the next result B^c stands for the completion of an algebra B and the inequality $u \ll w$ means that $u, w \in B^c$ and for every ultrafilter $F \subset B$ such that $x \wedge u \neq 0$ for every $x \in F$, there exists $y \in F$ such that $y \leqslant w$.

<u>Theorem</u> 2. If B is an infinite Boolean algebra and L is an upward σ-complete sublattice of B^c such that $B \subset L \subset B^c$ and for every $u \in L$ there exists $\{u_n : n < \omega\} \subset L$ such that $\inf\{u \wedge u_n : n < \omega\} = 0$, $u \vee u_n = 1$ and $u_{n+1} \ll u_n$ for every $n < \omega$, then $|L|$ is an ω-power.

The next result shows that the assumption that L is upward σ-complete and $B \subset L \subset B^c$ does not suffices for proving in ZFC that $|L|$ is an ω-power. Nomely, we have

<u>Theorem</u> 3. If $2^{\aleph_n} = \aleph_{\omega+n+1}$ for every $n<\omega$, then there exists an infinite Boolean algebra B and an upward σ-complete sublattice L of B^c such that $B\subset L\subset B^c$ and $|L|$ is not ω-power.

The aim of this note is to show that under the assumption of generalized continuum hypothesis (GCH) the situation is quite different. To do this I shall adapt an idea due to S.Koppelberg [5].

<u>Theorem</u> 4. Assume GCH. If B is an infinite Boolean algebra and L is an upward σ-complete sublattice of B^c such that $B\subset L\subset B^c$, then $|L|$ is an ω-power.

Proof. Let $\beta = |B| \geqslant \omega$. Since $|B^c| \leqslant 2^{|B|} = \beta^+$, the power of L equals either β or β^+. Clearly, we may assume that $|L| = \beta$ and β is a limit cardinal, i.e. $\beta = \sup\{\beta_\xi : \xi < cf(\beta)\}$, where $\beta_\xi < \beta_\eta < \beta$ for every $\xi<\eta<cf(\beta)$. If $cf(\beta) > \aleph_0$, then by Tarski's formula, we get

$$\beta^{\aleph_0} = (\sup\{\beta_\xi^+ : \xi < cf(\beta)\})^{\aleph_0} = \sup\{(\beta_\xi^+)^{\aleph_0} : \xi < cf(\beta)\} = \beta.$$

So, it remains to show that $cf(\beta) > \aleph_0$. Assume the contrary : $\beta = \sup\{\beta_n : n<\omega\}$, where $\beta_n < \beta_k < \beta$ for every $n<k<\omega$. Let $L = \{u_\xi : \xi < \beta\}$ and $L_n = L\cap B_n$, where B_n is a subalgebra of B^c generated by the set $\{u_\xi : \xi < \beta_n\}$. Then every L_n is a sublattice of L and it has the following property :

(1) if $u\in L_n$ and $-u\in L$, then $-u\in L_n$.

Now, for every $u\in L$ we define

$$i(u) = \min\{i : u\in L_i\}.$$

Since $L = \bigcup\{L_n : n<\omega\}$, the index $i(u)$ is well defined for every $u\in L$. Condition (1) follows that $i(u) = i(-u)$ for every $u\in B$; recall that $B\subset L$. We define by induction a sequence $\{z_n : n<\omega\}\subset B$ such that

(2) $0 < z_{n+1} < z_n$ for every $n<\omega$,

(3) $n<p$ implies $i(z_n) < i(z_p)$,

(4) for every $n<\omega$, $|B\restriction z_n| = \beta$,

where $B\restriction z = \{x\in B : x\leqslant z\}$. Assume $z_0,\dots,z_n$ are just defined. Since $|L_{i(n)}| \leqslant \beta_{i(n)} < \beta$ and $|B\restriction z_n| = \beta$, there exists $x\in B\restriction z_n$ such that $x\in L_{i(z_n)}$. Since the sequence $\{L_n : n<\omega\}$ is increasing we get

$$0 < x < z_n \text{ and } i(z_n) < i(x).$$

If $|B\restriction x| = \beta$, we set $z_{n+1} = x$. If not, then $|B\restriction z_n - x| = \beta$ and we set $z_{n+1} = z_n - x$. Since $i(u) = i(-u)$ for every $u\in B$ and $-x = -z_n \vee (z_n - x)$, $i(z_{n+1}) = i(x) > i(z_n)$. Now, for every $n<\omega$ we set

$$u_n = z_n - z_{n+1}.$$

The sequence $\{u_n : n<\omega\}$ consists of non-zero disjoint elements of B and, by the condition (3), $i(u_n) = i(z_{n+1})$ for every $n < \omega$. Hence, the set $N = \{i(u_n) : n < \omega\}$ is infinite. There exist infinite pairwise disjoint sets N_k such that $N = \bigcup\{N_k : k < \omega\}$. Since the lattice L is upward σ-complete, for every $k < \omega$ there exists an element $s_k \in L$ such that

$$s_k = \sup\{u_n : i(u_n) \in N_k\} .$$

The set $\{s_k : k < \omega\}$ consists of disjoint elements and for every $k < \omega$ there exists $u_{n(k)} < s_k$ such that

$$(5) \quad i(u_{n(k)}) \geqslant \max\{k, i(s_k)\} ,$$

which follows from the fact that every set N_k contains arbitrary large indexes. Now, we set

$$w = \sup\{u_{n(k)} : k < \omega\} .$$

Note that $w \wedge s_k = u_{n(k)}$. Hence, by condition (5), $i(w) \geqslant i(u_{n(k)})$ for every $k < \omega$. Then, again by the condition (5), $i(w) \geqslant k$ for every $k < \omega$, which is absurd. The proof is complete.

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