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In: Zdeněk Frolík and Vladimír Souček and Marián J. Fabián (eds.): Proceedings of the 14th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 14. pp. [305]–307.

Persistent URL: http://dml.cz/dmlcz/701903

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ON THE CLARKE'S GENERALIZED JACOBIAN

M. Fabian and D. Preiss

Let $f \colon \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a locally Lipschitz function defined on an open ball $B(x,r) \subset \mathbb{R}^n$ centered at x and of radius r > 0. According to the Rademacher's theorem [5] there exists a set $E_0 \subset \mathbb{R}^n$ of Lebesgue measure zero such that the Gâteaux derivative Df(y) exists whenever $y \in B(x,r) \setminus E_0$. Using this fact Clarke [2] introduced the generalized Jacobian $\partial f(x)$ as the closed convex hull of all possible limits $\lim_{y_1 \to y} Df(y_1)$, where $y_1 \in B(x,r) \setminus E_0$. Similarly, if E_0 is replaced by a null set $E \subset \mathbb{R}^n$ containing E_0 , one can define $\partial_E f(x)$. Thus $\partial_{E_0} f(x) = \partial f(x)$. For k = 1 Clarke [1] showed that $\partial_E f(x) = \partial f(x)$ for any null set E containing E_0 and asked in [2] if the equality remains true for k > 1. In what follows we answer this question affirmatively by showing

Theorem. $\partial_{\mathbf{E}} f(\mathbf{x}) = \partial f(\mathbf{x})$ for all k and for all null sets E including E₀.

<u>Proof.</u> All the spaces R^m are considered with the Euclidean norm $\|\cdot\|$. The symbol $\langle\cdot,\cdot\rangle$ denotes the usual inner product. The space of linear mappings from R^n to R^k as well as its dual will be identified with R^{nk} . Since clearly $\partial_E f(x) \subset \partial f(x)$, it remains to prove the converse. By contradiction, let us assume that this inclusion is proper. Then there exist a functional A in R^{nk} and $\alpha \in R$ such that

 $\sup \left\{ \langle A,L \rangle \colon \ L\epsilon \ \partial_E f(x) \right\} < \omega < \sup \left\{ \langle A,L \rangle \colon \ L\epsilon \ \partial f(x) \right\} \ .$ The definition of $\partial_E f(x)$ yields an $\epsilon > 0$ such that $\langle A,Df(y) \rangle < \omega \quad \text{whenever} \quad y \in B(x,\epsilon) \setminus E \ .$

Indeed, otherwise we could find $y_i \in B(x,1/i) \setminus E$ with

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 $\langle A, Df(y_i) \rangle \geqslant \lambda$ and hence there would exist an $L \in \partial_E f(x)$ such that $\langle A, L \rangle \geqslant \lambda$, which is impossible. Also, according to the definition of $\partial f(x)$, there is $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n) \in B(x, \varepsilon) \setminus E_0$ with $\alpha \leqslant \langle A, Df(\bar{y}) \rangle$.

By joining the last two inequalities we get

 $\langle A, Df(y) \rangle < \alpha < \langle A, Df(\bar{y}) \rangle$ whenever $y \in B(x, \varepsilon) \setminus E$. Denoting $A = (a_{i,j})$, i = 1, ..., k, j = 1, ..., n, and $g_j = a_{i,j} f_i + a_{i,$

+ ... + $a_{k,j}f_k$, j = 1,...,n, we can write the above inequality in the form

$$\sum_{j=1}^{n} \frac{\partial g_{j}(y)}{\partial y_{j}} < \alpha < \sum_{j=1}^{n} \frac{\partial g_{j}(\bar{y})}{\partial y_{j}} \quad \text{whenever} \quad y \in B(x, \epsilon) \setminus E.$$

Let C(s) be the n-dimensional cube with apices $(\bar{y}_1 \pm s, \dots, \bar{y}_n \pm s)$. Whenever s > 0 is so small that $C(s) \subset B(x, \epsilon)$, the above inequality holds almost everywhere in C(s) and, consequently,

Let us denote

 $\mathbf{S}_{\mathbf{j}}(\mathbf{s}) = \sup \left\{ \left| \mathbf{g}_{\mathbf{j}}(\mathbf{y}) - \mathbf{g}_{\mathbf{j}}(\mathbf{\bar{y}}) - \mathbf{D}\mathbf{g}_{\mathbf{j}}(\mathbf{\bar{y}})(\mathbf{y} - \mathbf{\bar{y}}_{\mathbf{j}}) \right| : \max_{\mathbf{i} = 1, \dots, n} \left| \mathbf{y}_{\mathbf{i}} - \mathbf{\bar{y}}_{\mathbf{i}} \right| \leq \mathbf{s} \right\},$ and

$$C_{j}(s) = \{(y_{1}, \dots, y_{j-1}, y_{j+1}, \dots, y_{n}): (y_{1}, \dots, y_{j-1}, \overline{y}_{j}, y_{j+1}, \dots, y_{n}) \in C(s)\}, \quad j = 1, \dots, n, \quad s > 0.$$
 Using the Fubini theorem, we get

$$\int_{C(s)}^{s} \frac{\partial g_{j}(y)}{\partial y_{j}} dy_{1} ... dy_{n} =$$

$$= \int_{C_{j}(s)}^{s} \sum_{\sigma = \pm 1}^{\sigma} \sigma g_{j}(y_{1}, ..., y_{j-1}, \overline{y}_{j} + \overline{\sigma} s, y_{j+1}, ..., y_{n}) \times$$

$$\times dy_{1} ... dy_{j-1} dy_{j+1} ... dy_{n} =$$

$$= \int_{C_{j}(s)} \left(\sum_{\sigma = \pm 1} \sigma \left[g_{j}(y_{1}, \dots, y_{j-1}, \overline{y}_{j} + \overline{v}_{s}, y_{j+1}, \dots, y_{n}) - g_{j}(\overline{y}) - Dg_{j}(\overline{y})(y_{1} - \overline{y}_{1}, \dots, y_{j-1} - \overline{y}_{j-1}, \sigma_{s}, y_{j+1} - \overline{y}_{j+1}, \dots, y_{n} - \overline{y}_{n}) \right] + 2s \frac{\partial g_{j}(\overline{y})}{\partial y_{j}} \times dy_{1} \dots dy_{j-1} dy_{j+1} \dots dy_{n} \ge -2(2s)^{n-1} \delta_{j}(s) + 2s \frac{\partial g_{j}(\overline{y})}{\partial y_{j}} (2s)^{n-1} .$$

Hence (*) implies

$$(2s)^n \sum_{j=1}^n \frac{\partial g_j(\bar{y})}{\partial y_j} - (2s)^n \sum_{j=1}^n \delta_j(s)/s < (2s)^n \alpha < (2s)^n \sum_{j=1}^n \frac{\partial g_j(\bar{y})}{\partial y_j}$$
 if $s > 0$ is sufficiently small. Let us note that $\delta_j(s)/s \longrightarrow 0$ as $s \downarrow 0$ since Gâteaux and Fréchet differentiability in finite-dimensional spaces coincide for Lipschitz functions. Thus, dividing the above inequality by $(2s)^n$ and letting s go to zero, we obtain a wrong inequality. This contradiction finishes the proof.

Remark 1. For f and x as above Pourciau [4] considered a generalized Jacobian which in our notation is equal to $\partial_{E_1} f(x)$ with $E_1 = E_0 \bigcup \left\{ y \in B(x,r) \setminus E_0 : y \text{ is not a Lebesgue point of Df} \right\}$. As f is locally Lipschitz, E_1 is a null set. Hence by Theorem $\partial_{E_1} f(x) = \partial f(x)$.

Remark 2. The reader probably noticed that the above proof is actually based on the Gauss - Green theorem. In fact, this theorem shows a "Denjoy property for derivatives of mappings between \mathbb{R}^n and \mathbb{R}^k " suggested by [2, Remark 5].

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