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Let $\mathbb{C}$ be a class of objects with a binomial coefficient $\mathbb{C}\binom{A}{B}$. Intuitively $\mathbb{C}\binom{A}{B}$ is the set of all embeddings of B into A, respectively the set of all subobjects of A which are isomorphic to B. Embeddings $f \in \mathbb{C}\binom{A}{B}$ and $g \in \mathbb{C}\binom{B}{C}$ may be composed yielding $fg \in \mathbb{C}\binom{A}{C}$. As known for categories this composition should be associative.

<u>Notation:</u> $\Pi(\mathbb{C}\binom{A}{C})$ denotes the set of equivalence relations on $\mathbb{C}\binom{A}{C}$. For $\pi \in \Pi(\mathbb{C}\binom{A}{C})$ and $f \in \mathbb{C}\binom{A}{B}$ then $\pi_f \in \Pi(\mathbb{C}\binom{B}{C})$ denotes the restriction of π to the subobjects of f, i.e. $g \approx h (\mathrm{mod}\ \pi_f)$ iff $fg \approx fh (\mathrm{mod}\ \pi)$.

<u>Definition:</u> A set $\mathcal{A} \subset \Pi(\mathbb{C}\binom{B}{C})$ is a canonical set of equivalence relations (w.r.t. B and C) iff (1) there exists an $A' \in \mathbb{C}$ such that for all $A \in \mathbb{C}$ with $\mathbb{C}\binom{A}{A'} \neq \emptyset$ and $\pi \in \Pi(\mathbb{C}\binom{A}{C})$ there exists an $f \in \mathbb{C}\binom{A}{B}$ such that $\pi_f \in \mathcal{A}$ and (2) for every $\sigma \in \mathcal{A}$ and every $A \in \mathbb{C}$ there exists a $\pi \in \Pi(\mathbb{C}\binom{A}{C})$ such that $\pi_f = \sigma$ for every $f \in \mathbb{C}\binom{A}{B}$.

<u>Motivation:</u> In recent years it has turned out that canonical sets of equivalence relation yield a deeper insight into the partitional behaviour of certain structures. One of the first theorems in this

direction is the 'Erdös-Rado-canonization theorem' [1] which describes canonical equivalence relations for Ramsey's theorem. More recent results are due to A. Taylor [4] for Hindman's theorem and Nešetřil and Rödl [5] for graphs and hypergraphs.

Here we study Hales-Jewett classes $[A]$, where A is a finite set.

Definition: Let A be a finite set, $k \leq n$ be non-negative integers. $[A]\binom{n}{k}$ than is the set of mappings $f: n \to A \cup \{\lambda_0, \ldots, \lambda_{k-1}\}$ - where for convenience always $n = \{0, \ldots, n-1\}$ - satisfying
(1) $f^{-1}(\lambda_i) \neq \emptyset$ for $i < k$ and (2) $\min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_j)$ for all $i < j < k$.

Parameter words $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$ may be composed yielding $f \cdot g \in [A]\binom{n}{k}$, where $f \cdot g(i) = f(i)$ for $f(i) \in A$ and $f \cdot g(i) = g(j)$ for $f(i) = \lambda_j$.

Notations: For sets $A \subseteq B$ we always assume $\Pi(A) \subseteq \Pi(B)$ by extending $b \approx b (\mathrm{mod}\ \pi)$ for all $b \in B \setminus A$, $\pi \in \Pi(A)$. $\Pi(A)$ is partially ordered (in fact it is a lattice) by $\pi \leq \sigma$ iff $a \approx b (\mathrm{mod}\ \pi)$ implies $a \approx b (\mathrm{mod}\ \sigma)$ for all $a, b \in A$.

Definition: A sequence $\hat{\pi} = (\pi_0, \ldots, \pi_k)$ of equivalence relations $\pi_i \in \Pi(A \cup \{\lambda_0, \ldots, \lambda_i\})$ is k-canonical iff (1) $\pi_0 \leq \pi_1 \leq \ldots \leq \pi_k$ and (2) if $\lambda_i \approx a (\mathrm{mod}\ \pi_i)$ for some $a \in A \cup \{\lambda_0, \ldots, \lambda_{i-1}\}$ then $\pi_{i+1} \leq \pi_i$.

__Notation:__ Let $\hat{\pi} = (\pi_0,\dots,\pi_k)$ be k-canonical and $f \in [A]\binom{n}{k}$. For $i < k$ the numbers $\omega_{\hat{\pi}}(f,i)$ are defined by $\omega_{\hat{\pi}}(f,i) = \mathrm{MIN}\{\xi > \omega_{\hat{\pi}}(f,i-1) \mid f(\xi) \approx \lambda_i \pmod{\pi_i}\}$, where for convenience $\omega_{\hat{\pi}}(f,-1) = -1$.

__Definition:__ Let $\hat{\pi} = (\pi_0,\dots,\pi_k)$ be k-canonical. The partition $\hat{\pi}(n) \in \Pi([A]\binom{n}{k})$ is defined by:

$$f \approx g \bmod(\hat{\pi}(n)) \text{ iff for all } i = -1\ 0,\dots,k-1 \text{ for all } \xi \text{ with}$$
$$\mathrm{MIN}(\omega_{\hat{\pi}}(f,i), \omega_{\hat{\pi}}(g,i)) < \xi < n \text{ follows } f(\xi) \approx g(\xi) \pmod{\pi_{i+1}}.$$

The main theorem then is:

__Theorem:__ The set $\{\hat{\pi}(m) \mid \hat{\pi}$ is k-canonical$\}$ is a canonical set of equivalence relations (w.r.t. $[A],k,m$).

The theorem has many interesting applications, we only state two of them explicitly.

__Corollary 1:__ For every finite set A and non-negative integer m there exists an n, such that for every equivalence relation $\sigma \in \Pi([A]\binom{n}{0})$ there exists an $f \in [A]\binom{n}{m}$ and a $\pi \in \Pi(A)$ such that for all $g,h \in [A]\binom{m}{0}$ follows $g \approx h \pmod{\pi_f}$ iff $g(\xi) \approx h(\xi) \pmod{\sigma}$ for every $\xi < m$.

__Corollary 2:__ For every positive integer m there exists an n such that for every equivalence relation $\pi \in \Pi(\binom{P(n)}{P(1)})$ on the set of pairs (A,B) of subsets of n with $A \subseteq B$, $A \neq B$ there exists a $P(m)$-sublattice of $P(n)$, i.e. sets $X_0, X_1, \ldots, X_m$, with $X_i \cap X_j = X_0$ and $X_i \neq X_j$ for $i \neq j$ such that one of the following 5 case holds for all pairs (A,B), (C,D) $A \subsetneqq B$, $C \subsetneqq D$: .

(1) $(A,B) \approx (C,D) \pmod{\pi}$ iff $A = C$ and $B = D$

(2) $(A,B) \approx (C,D) \pmod{\pi}$ iff $B \setminus A = D \setminus C$

(3) $(A,B) \approx (C,D) \pmod{\pi}$ iff $A = C$

(4) $(A,B) \approx (C,D) \pmod{\pi}$ iff $B = D$

(5) $(A,B) \approx (C,D) \pmod{\pi}$ for all (A,B), (C,D).

Analogous theorems may be established for finite vector spaces and affine spaces. Details and proofs are going to appear elsewhere.

References

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